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Duality-Invariant Orthogonality Relation Between the Magnetic Monopole and Associated Quark Charges

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Duality invariance of the Dirac–Schwinger charge-symmetric theory for electromagnetism leads one to consider the complex-valued amplitudes ψ_1 and ψ_2 for the separation between the magnetic monopole and quarks in the logarithmic charge plane. It is observed that the orthogonality relation on the latter amplitudes, $\mathrm{Re}(\psi_1^* \psi_2) = 0$, is equivalent to the equation $(\ln 9\alpha^{-1})(\ln 2) = (1/2)\pi^2$, which is indeed satisfied by the experimental value for α to within 0.027%. In addition to fixing the unit of electric charge at a primary physical value, the orientation of ψ_1, ψ_2 may also prescribe the Cabibbo angle to have the theoretical value 12.4438°.

1. INTRODUCTION

A beautiful feature of the Dirac–Schwinger charge-symmetric theory for electromagnetism is its invariance under *duality tranformations* (Dirac, 1934, 1948; Schwinger, 1966)

$$\Phi_{\mu\nu} \to \Phi'_{\mu\nu} = e^{i\theta}\Phi_{\mu\nu} \tag{1}$$

where the duality angle θ is a disposable parameter and

$$\Phi_{\mu\nu} \equiv F_{\mu\nu} + \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{2}$$

is the linear combination of the electromagnetic tensor $F_{\mu\nu} \equiv -F_{\nu\mu}$ and $i \equiv (-1)^{1/2}$ times its Levi-Cività dual. All electric and magnetic coupling-constant charges transform concomitantly with (1), viz.

$$\zeta_k \to \zeta'_k = e^{i\theta}\zeta_k, \qquad \zeta_k \equiv e_k + ig_k \tag{3}$$

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for the kth field with particles carrying electric charge e_k and magnetic charge g_k. The Dirac–Schwinger coexistence constraints on the latter constants for the kth and lth charge-carrying fields,

$$e_k g_l - g_k e_l = \mathrm{Im}(\zeta_k^* \zeta_l) = 4\pi n_{kl} \tag{4}$$

with $n_{kl} \equiv -n_{lk}$ an array of integers, are clearly invariant under (3), as are the electromagnetic field equations, the particle-motion equations, and the quantum commutation relations for electromagnetic radiation (see Appendix A). "Purely electric" charges observed to date lie on a straight line through the origin in the complex ζ plane; however, it is merely by convention that this straight line is chosen to be the real axis (so that $\mathrm{Im}\,\zeta_k = 0$ for all k), in light of the duality invariance of the theory.

Whereas the ζ plane is rotated by angle θ under a duality transformation, a linear translational symmetry is induced in the $\ln \zeta$ plane, for it follows from (3) that

$$\ln \zeta_k \to \ln \zeta_k' = \ln \zeta_k + i\theta \tag{5}$$

Owing to the linear translational form of (5), special physical significance may be attached to the duality-invariant quantities $(\ln \zeta_k - \ln \zeta_l)$ for each pair of complex charges ζ_k and ζ_l. In particular, for the quark fields with $e_1 = -e/3$ and $e_2 = 2e/3$ and the associated magnetic monopole field with $g_3 = -12\pi/e$ [the minimal integer value $n_{13} = 1$ requires $e_1 g_3 = 4\pi$ in (4), and therefore $g_3 = -12\pi/e$ for $e_1 = -e/3$] one has

$$\zeta_1 = -e/3, \qquad \zeta_2 = 2e/3, \qquad \zeta_3 = -12\pi i/e \tag{6}$$

The related logarithms are (see Appendix B for the charge-conjugate expressions)

$$\ln \zeta_1 = \ln(e/3) + i\pi, \qquad \ln \zeta_2 = \ln(2e/3), \qquad \ln \zeta_3 = \ln(12\pi/e) + 3i\pi/2 \tag{7}$$

and thus the duality-invariant quantities follow as

$$\begin{aligned} \psi_1 &\equiv \ln \zeta_3 - \ln \zeta_1 = \ln(9\alpha^{-1}) + i\pi/2 \\ \psi_2 &\equiv \ln \zeta_1 - \ln \zeta_2 = -\ln 2 + i\pi \end{aligned} \tag{8}$$

2. THE ORTHOGONALITY RELATION

As suggested by the kernel symbol ψ, the quantities (8) are complex-valued amplitudes for the separation between the magnetic monopole and quarks in the logarithmic charge plane: $|\psi_1| = [(\ln 9\alpha^{-1})^2 + \pi^2/4]^{1/2}$ is the separation between the quark charge $-e/3$ and the associated magnetic monopole charge, while $|\psi_2| = [(\ln 2)^2 + \pi^2]^{1/2}$ is the separation between the quark charges $-e/3$ and $2e/3$. The amplitude for the separation between quark charge $2e/3$ and the magnetic monopole charge follows from (7) and (8) as

$$\psi_3 = \ln \zeta_3 - \ln \zeta_2 = \psi_1 + \psi_2 \tag{9}$$

and therefore

$$|\psi_3|^2 = |\psi_1|^2 + 2\,\mathrm{Re}(\psi_1^* \psi_2) + |\psi_2|^2 \tag{10}$$

For orthogonality between the amplitudes ψ_1 and ψ_2,

$$\mathrm{Re}(\psi_1^* \psi_2) = 0, \tag{11}$$

(10) reduces to the right-triangle relation $|\psi_3| = (|\psi_1|^2 + |\psi_2|^2)^{1/2}$. By substituting the amplitudes (8) into (11), one finds that the orthogonality condition holds if and only if

$$(\ln 9\alpha^{-1})(\ln 2) = \pi^2/2 \tag{12}$$

The left and right sides of (12) equal 4.933453 and 4.934802, respectively, and thus (12) is indeed satisfied by the experimental value $\alpha^{-1} = 137.036$ to within 0.027%. Hence the orthogonality condition (11) between the magnetic monopole and associated quark charges is valid empirically to high accuracy. Conversely, a primary theoretical value for the electric charge unit e is obtainable by assuming that (11) is valid precisely in basic theory. One obtains (12) from (11) and the latter equation yields $e = (4\pi\alpha)^{1/2} = 6\pi^{1/2} \exp(-\pi^2/4 \ln 2) = 0.3025276$, which is about 0.097% less than the experimental value. This may signify a small finite (extractable) correction in the charge renormalization of quark electrodynamics.

3. POSSIBLE CONNECTION WITH THE CABIBBO ANGLE

The orthogonality condition (11) states that ψ_1 and ψ_2 differ in phase by 90°. Since (11) is slightly imprecise, the phase angles obtained from (8)

$$\begin{aligned} \theta_1 &\equiv \arg \psi_1 = \tan^{-1}\left[\frac{\pi/2}{\ln(9\alpha^{-1})}\right] = 12.4454° \\ \theta_2 &\equiv \arg \psi_2 = 90° + \tan^{-1}\left(\frac{\pi}{\ln 2}\right) = 90° + 12.4422° \end{aligned} \tag{13}$$

differ by 90° minus the small angle $0.0032° = 11.52''$. However, the near perpendicularity of ψ_1 and ψ_2 and their mutual "tilt" in the plane by $\bar{\theta} \pm 0.0016°$, where

$$\bar{\theta} \equiv \tfrac{1}{2}(\theta_1 + \theta_2 - 90°) = 12.4438° \tag{14}$$

suggests that the Cabibbo (1963) angle $\theta_c \cong 12.5°$ may be given theoretically by $\theta_c = \bar{\theta}$. As is well known, the quark states u, d, c, and s are the degrees of freedom appropriate to the flavor-preserving strong interaction, while the states u, d', c, and s' with the Cabibbo rotation

$$\begin{aligned} |d'\rangle &\equiv (\cos\theta_c)|d\rangle + (\sin\theta_c)|s\rangle \\ |s'\rangle &\equiv (\cos\theta_c)|s\rangle - (\sin\theta_c)|d\rangle \end{aligned} \tag{15}$$

are the "normal mode" degrees of freedom for the charge-changing weak interaction. The direction-preserving association

$$\begin{aligned} |d'\rangle &\leftrightarrow \psi_1/|\psi_1|, \qquad |d\rangle \leftrightarrow 1 \\ |s'\rangle &\leftrightarrow \psi_2/|\psi_2|, \qquad |s\rangle \leftrightarrow i \end{aligned} \tag{16}$$

may result from the same mechanism that is responsible for condition (11).

APPENDIX A: DUALITY INVARIANCE OF THE FIELD AND PARTICLE-MOTION EQUATIONS

If the quanta of the fermion field Ψ_k bear electric and magnetic charge e_k and g_k, then the associated field equations $\partial_\nu \Phi^{\mu\nu} = \sum_k \zeta_k \bar{\Psi}_k \gamma^\mu \Psi_k$ are clearly invariant, with (1) and (3) producing the factor $e^{i\theta}$ on both sides. In terms of (2) and the complex charge $\zeta_k \equiv e_k + ig_k$, the classical equation of particle motion $m_k d^2x^\mu/d\tau^2 = \mathrm{Re}(\zeta_k^* \Phi^{\mu\nu})\, dx_\nu/d\tau$ is likewise invariant under the duality transformations (1) and (3). Similarly, for a fermion field in the quantum theory one has the duality-invariant Dirac field equation $\gamma^\mu[\partial_\mu + i\,\mathrm{Re}(\zeta_k^* \rho_\mu)]\Psi_k = m_k \Psi_k$, where ρ_μ is the complex potential associated with (2): $\Phi_{\mu\nu} = \partial_\mu \rho_\nu - \partial_\nu \rho_\mu$, $\rho'_\mu = e^{i\theta}\rho_\mu$. Finally, duality invariance of the equal-time commutation relations for electromagnetic radiation becomes explicitly evident if the latter relations are expressed in terms of (2), viz. $[\Phi_{0j}^{\mathrm{tr}}(\mathbf{x}), \Phi_{0k}^{\mathrm{tr}}(\mathbf{x}')^*] = 2\varepsilon_{jkm}\partial_m \delta(\mathbf{x}-\mathbf{x}')$, $[\Phi_{0j}^{\mathrm{tr}}(\mathbf{x}), \Phi_{0k}^{\mathrm{tr}}(\mathbf{x}')] = 0 = [\Phi_{0j}^{\mathrm{tr}}(\mathbf{x})^*, \Phi_{0k}^{\mathrm{tr}}(\mathbf{x}')^*]$, where Φ_{0j}^{tr} are the components of the transverse part of the 3-vector; thus $\Phi_{0j}^{\mathrm{tr}} \to e^{i\theta}\Phi_{0j}^{\mathrm{tr}}$, $\Phi_{0j}^{\mathrm{tr}*} \to e^{-i\theta}\Phi_{0j}^{\mathrm{tr}*}$.

APPENDIX B: CHARGE-CONJUGATION EQUIVALENCE FORMULAS

In (7) one takes the branch of $\ln\zeta$ for which $0 \le \mathrm{Im}(\ln\zeta) < 2\pi$ with the cut in the ζ plane running just below the positive real axis from $\zeta = 0$ to $\zeta = +\infty - i\varepsilon$. If the charges conjugate to (6) are considered, namely, $\bar{\zeta}_1 = +e/3$, $\bar{\zeta}_2 = -2e/3$, $\bar{\zeta}_3 = +12\pi i/3$, then the appropriate branch for the $\ln\bar{\zeta}$ derives from the ζ plane under the mapping $\bar{\zeta} \equiv -\zeta$. Hence the cut in the $\bar{\zeta}$ plane runs just above the negative real axis from $\bar{\zeta} = 0$ to $\bar{\zeta} = -\infty + i\varepsilon$, and one has $-\pi \leq \mathrm{Im}(\ln \bar{\zeta}) < \pi$. It follows that

$$\bar{\psi}_1 \equiv \ln \bar{\zeta}_3 - \ln \bar{\zeta}_1 = \ln 9\alpha^{-1} + i\pi/2$$

$$\bar{\psi}_2 \equiv \ln \bar{\zeta}_1 - \ln \bar{\zeta}_2 = -\ln 2 + i\pi$$

i.e., the same values as shown in (8), and thus one finds that the conjugate version of (11), $\mathrm{Re}(\bar{\psi}_1^* \bar{\psi}_2) = 0$, is also equivalent to (12).

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